SOME MOMENT INEQUALITIES FOR STOCHASTIC INTEGRALS AND FOR SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT

Some inequalities concerning the Itô stochastic integral and solutions of stochastic different equations are obtained.

1. Stochastic integrals. Let (Ω, B, P) be a probability space, and $w(t,\omega)$, $t \in [0,T]$, $\omega \in \Omega$, a standard Wiener process. Let B_t, $t \in [0,T]$ be an increasing system of sub-σ-fields of B, such that for all $t \in [0,T]$, $w(s,\omega)$ $(0 \leqq s \leqq t)$ is measurable with respect to B_t, and such that if $t_1, t_2, \cdots, t_k \in [t,T]$, then the aggregate of differences $w(t_j,\omega) - w(t,\omega)$, $j = 1,2,\cdots,k$ is independent of B_t. Finally, let $f(t,\omega)$ be a real-valued measurable random function, such that for each t in $[0,T]$, $f(t,\omega)$ is measurable with respect to B_t and for almost all $\omega \in \Omega$

$$\int_0^T f^2(t,\omega)dt < \infty.$$

Under these assumptions, the Itô stochastic integral ([1], [2])

$$\int_0^T f(t,\omega)dw(t,\omega)$$

exists, and if

$$E\int_0^T f^2(t,\omega)dt < \infty$$

then

$$E\int_0^T f(t,\omega)dw(t,\omega) = 0$$

$$E\left(\int_0^T f(t,\omega)dw(t,\omega)\right)^2 = E\int_0^T f^2(t,\omega)dt.$$

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REMARK. $E^p(\ )$ will be used as an abbreviation for $[E(\ )]^p$.

THEOREM 1. *If, for some $p>2$, $\int_0^T E^{2/p}|f(t,\omega)|^p dt<\infty$, then*

$$E\left|\int_0^T f(t,\omega)dw(t,\omega)\right|^p<\infty \tag{1}$$

and

$$E^{2/p}\left|\int_0^T f(t,\omega)dw(t,\omega)\right|^p \leqq (p-1)\int_0^T E^{2/p}|f(t,\omega)|^p dt.$$

In order to prove (1) we first prove the following

LEMMA: *If $\phi(s)$ is non-negative and continuous; $\psi(s)$ is non-negative, Lebesgue-measurable, $\int_0^T\psi(s)ds<\infty.$, and if for $0\leqq t\leqq T$ and $0<\alpha<1$*

$$\phi(t)\leqq \delta+\int_0^t \psi(s)(\phi(s))^{1-\alpha}ds, \qquad \delta>0, \tag{2}$$

then

$$\phi(t)\leqq \left(\delta^\alpha+\alpha\int_0^t\psi(s)ds\right)^{1/\alpha}. \tag{3}$$

Proof of the lemma. Under the additional assumption that $\psi(t)$ is continuous it follows from (2) that

$$\frac{d\left(\delta+\int_0^\theta \psi(s)\phi^{1-\alpha}(s)ds\right)^\alpha}{\alpha d\theta}=\frac{\psi(\theta)\phi^{1-\alpha}(\theta)}{\left(\delta+\int_0^\theta\psi(s)\phi^{1-\alpha}(s)ds\right)^{1-\alpha}}\leqq\psi(\theta). \tag{4}$$

The above inequality can be integrated to obtain

$$\left(\delta+\int_0^t\psi(s)\phi^{1-\alpha}(s)ds\right)^\alpha\leqq\delta^\alpha+\alpha\int_0^t\psi(\theta)d\theta \tag{5}$$

and (3) follows from (5) and (2). The additional restriction that $\psi(s)$ be continuous can be removed, since for almost all θ in $[0,T]$, $\delta+\int_0^\theta\psi(s)\phi^{1-\alpha}(s)ds$ is differentiable, with $\psi(\theta)\phi^{1-\alpha}(\theta)$ as its derivative. For each such θ, (4) is true. Since $(\delta+\int_0^\theta\psi\phi^{1-\alpha})^\alpha$ is absolutely continuous, it is the integral of its almost everywhere derivative; therefore, (5) holds and (3) follows.

From now on the variable ω will be omitted.

Proof of Theorem 1. We assume first that a.s. $|f(t)|\leqq k$ for all t in $[0,T]$. It follows, then, from a theorem of Dynkin ([2], Theorem 7.3) that

$$E\left\{\exp \pm \int_0^T f(s)dw(s)\right\} \leqq \exp k^2 T$$

and therefore all the moments of $\int_0^T f(s)dw(s)$ are finite. For any $\delta > 0$, $\left|\int_0^T f(s)dw(s)\right|^p \leqq (\delta + (\int_0^T f(s)dw(s))^2)^{p/2}$, and by Ito's formula [3]

$$\left|\delta + \left(\int_0^T f(s)dw(s)\right)^2\right|^{p/2} - \delta^{p/2}$$

$$= p\int_0^T \left[f(s)\left(\delta + \left(\int_0^s f(\zeta)dw(\zeta)\right)^2\right)^{p/2-1} \cdot \int_0^s f(\zeta)dw(\zeta)\right] dw(s)$$

$$+ \frac{1}{2}p\int_0^T \left[\left(\delta + \left(\int_0^s f(\zeta)dw(\zeta)\right)^2\right)^{p/2-1} f^2(s)\right] ds \tag{6}$$

$$+ \frac{1}{2}(p-2)p\int_0^T \left[\left(\delta + \left(\int_0^s f(\zeta)dw(\zeta)\right)^2\right)^{p/2-2} \left(\int_0^s f(\zeta)dw(\zeta)\right)^2 f^2(s)\right] ds.$$

The right hand side of (6) is of the form

$$\int_0^T F_1(s)dw(s) + \int_0^T F_2(s)ds. \tag{7}$$

Since $E|F_1(s)|^2$ and $E|F_2(s)|$ are bounded on $[0,T]$, the expectation of the first term of (7) is zero; and for the expectation of the second term the order of integration and expectation may be interchanged. Also, the expectation of the left hand side of (6) is continuous in t. Therefore

$$E\left(\delta + \left(\int_0^T f(s)dw(s)\right)^2\right)^{p/2} - \delta^{p/2} \leqq \frac{1}{2}(p^2-p)\int_0^T E\left\{f^2(s)\left(\delta + \left(\int_0^s f(\zeta)dw(\zeta)\right)^2\right)^{p/2-1}\right\} ds. \tag{8}$$

Applying to (8) the Hölder inequality with $\sigma = p/2$, $\rho = p/(p-2)$ we have

$$E\left(\delta + \left(\int_0^T f(s)dw(s)\right)^2\right)^{p/2} - \delta^{p/2}$$

$$\leqq \frac{1}{2}(p^2 - p)\int_0^T E^{2/p}|f(s)|^p E^{(p-2)/p}\left(\delta + \left(\int_0^s f(\zeta)dw(\zeta)\right)^2\right)^{p/2} ds.$$

By (2) and (3)

$$E\left(\delta + \left(\int_0^T f(s)dw(s)\right)^2\right)^{p/2} \leqq \left(\delta + (p-1)\int_0^T E^{2/p}|f(s)|^p ds\right)^{p/2}, \tag{9}$$

from which (1) follows after letting $\delta \to 0$.

In order to remove the additional restriction $|f(s)| \leqq k$, let $f_n(s) = \min[n, \max(-n, f(s))]$. Since $E^{1/p}|f(s)|^p$ is non-decreasing in p, the condition of Theorem 1 implies that $E\int_0^T f^2(t)dt < \infty$ and $\int_0^T f_n(s)dw(s)$ converges to $\int_0^T f(s)dw(s)$ in quadratic mean, Therefore there exists a subsequence n_i for which convergence is almost sure. For this subsequence we have

$$\left(\delta + (p-1)\int_0^T E^{2/p}\,|f(s)|\,ds\right)^{p/2} \geqq \varlimsup_{n_i\to\infty} E\left(\delta + \left(\int_0^T f_{n_i}(s)dw(s)\right)^2\right)^{p/2}$$
$$\geqq E\left(\delta + \left(\int_0^T f(s)dw(s)\right)^2\right)^{p/2}$$

where the first inequality follows from (9) and the second from Fatou's lemma.

COROLLARY 1. *Under the condition of Theorem* 1.

$$(10)\qquad E\left|\int_0^T f(s)dw(s)\right|^p \leqq (p-1)^{p/2}T^{(p-2)/2}\int_0^T E|f(s)|^p ds.$$

The proof follows from (9) by direct application of the Hölder inequality to $\int_0^T 1\, E^{2/p}|f(s)|^p ds$ with $\sigma = p/(p-2)$ and $\rho = p/2$.

REMARK: For $p = 4$, inequality (10) has been known ([1] p. 23, with 36 instead of $(p-1)^{p/2} = 9$, and [4] in a weaker form).

2. **Stochastic differential equations.** Let E_r be the Euclidean r-space. Let $m(t,x)$, $(t \geqq 0, x \in E_r)$ assume values in E_r; $G(t,x)$, $(t \geqq 0,\ x \in E_r)$ will assume real $r \times q$ matrix values and $w(t)$ will denote the q-dimensional Wiener process. The prime (′) will denote the transpose of a vector or a matrix; for vectors $|\cdot|$ will denote the Euclidean norm, for matrices G, $|G|$ will denote the norm $(\text{trace } GG')^{1/2}$ ([2] p. 209). Assume that $m(t,x)$ and $G(t,x)$ are measurable functions of their variables, that

$$(11)\qquad |m(t,x) - m(t,y)| \leqq k|x-y|, |G(t,x) - G(t,y)| \leqq k|x-y|,$$

and

$$\int_0^T (|m(t,0)|^2 + |G(t,0)|^2)dt < \infty.$$

Let a be a random variable assuming values in E_r and independent of $w(t)$, $t \geqq 0$. Under these assumptions, the stochastic differential equation

$$(12)\qquad dx(t) = m(t,x(t))dt + G(t,x(t))dw(t), \qquad x(0) = a$$

has a unique solution in $[0,T]$, and if $E|a|^2 < \infty$, then $E|x(t)|^2$ is bounded on $[0,T]$ ([1], [2]).

THEOREM 2. *If for some* $p > 2$

$$(13)\qquad \int_0^T (|m(t,0)|^p + |G(t,0)|^p)dt < \infty$$

and $E|a|^p < \infty$, *then* $E|x(t)|^p$ *is bounded on* $[0,T]$.

For $p = 2$ this result is usually obtained (together with the existence of solution) by the method of successive approximations ([1] p. 47, or [2] Theorem 11.1).

Using Corollary 1 (and the Holder inequality instead of the Schwarz inequality), the same proof applies directly to convergence in the mean of order p. The details are, therefore, omitted.

THEOREM 3. *If, for all* $t \geqq 0$.

$$|m(t,0)| + |G(t,0)| \leqq k_1 \tag{14}$$

then there exists a constant α *such that for all* $t \geqq 0$, *for all* a *in* E_r *and for all non-negative integers* n

$$E_a(1 + |x(t)|^2)^n \leqq (1 + |a|^2)^n e^{n(2n+1)\alpha t}. \tag{15}$$

If, in addition, $|G(t,x)| \leqq k_2$, $(t \geqq 0,\ x \in E_r)$, *then there exists a constant* β *such that for all* a *and for all* $t \geqq 0$

$$E_a(1 + |x(t)|^2)^n \leqq \frac{(2n)!}{n!2^n}\left(\frac{3}{2} + |a|^2\right)^n e^{2n\beta t}\ ; \tag{16a}$$

(which is the same as

$$E_a(1 + |x(t)|^2)^n \leqq E(\zeta(t))^{2n} \tag{16b}$$

where

$$\zeta(t) = \left(\frac{3}{2} + |a|^2\right)^{1/2} e^{\beta t}\zeta_0$$

and ζ_0 is a Gaussian random variable with $E\zeta_0 = 0$ and $E\zeta_0^2 = 1$).

Poorf. By Ito's formula ([2], [3]), for $n = 0, 1, 2, \cdots$

$$\begin{aligned}(1 + |x(t)|^2)^n = {} & (1 + |a|^2)^n + \int_0^t 2n(1 + |x(s)|^2)^{n-1} x'(s) m(t, x(s)) ds \\ & + \int_0^t 2n(1 + |x(s)|^2)^{n-1} x'(s) G(s, x(s)) dw(s) \\ & + \frac{1}{2}\int_0^t (1 + |x(s)|^2)^{n-1} 2n \cdot \text{trace}\,(G \cdot G') ds \\ & + \frac{1}{2}\int_0^t 4n(n-1)(1 + |x(s)|^2)^{n-2} x'(s) G G' x(s) ds.\end{aligned} \tag{17}$$

The right hand side of (17) is of the form

$$(1 + |a|^2)^n + \int_0^t F(s) dw(s) + \int_0^t f(s) ds.$$

By the result of Theorem 2, $E|F(s)|^2$ and $E|f(s)|$ are bounded on any bounded t interval. Therefore $E\int_0^t F(s)dw(s) = 0$ and $E_a(1 + |x(t)|^2)^n$ is continuous in t.

From (11) and (14) it follows that for some α

$$|m(t,x)| \leqq \alpha(1+|x|^2)^{1/2}$$
$$|G(t,x)|^2 = \operatorname{trace}(GG') \leqq \alpha(1+|x|^2).$$

Therefore, from (17),

$$E_a(1+|x(t)|^2)^n \leqq (1+|a|^2)^n + n(2n+1)\alpha\int_0^t E_a(1+|x(s)|^2)^n ds \tag{18}$$

Setting $E_a(1+|x(t)|^2)^n = \phi_n(t)$ and since $\phi_n(t)$ is continuous in t, we have from (18)

$$\frac{d\left\{e^{-n(2n+1)\alpha t}\int_0^t \phi_n(s)ds\right\}}{ds} \leqq \phi_n(0)e^{-n(2n+1)\alpha t}.$$

Integrating and substituting in (18), we get (15).

From the condition $|G| \leqq k_2$, (11) and (14) it follows that for some β

$$|m(t,x)| \leqq \beta(1+|x|^2)^{1/2}$$
$$|G(t,x)|^2 \leqq \beta.$$

Therefore, from (17), for $n = 1,2,\cdots$

$$\phi_n(t) \leqq \phi_n(0) + 2n\beta\int_0^t \phi_n(s)ds + n(2n-1)\beta\int_0^t \phi_{n-1}(s)ds, \tag{19}$$

or

$$\frac{d\left\{e^{-2n\beta t}\int_0^t \phi_n(s)ds\right\}}{dt} \leqq \phi_n(0)\,e^{-2n\beta t} + n(2n-1)\beta e^{-2n\beta t}\int_0^t \phi_{n-1}(s)ds.$$

Integrating and substituting in (19)

$$\phi_n(t) \leqq \phi_n(0)e^{2n\beta t} + \beta n(2n-1)e^{2n\beta t}\int_0^t e^{-2n\beta s}\phi_{n-1}(s)ds. \tag{20}$$

Repeated applications of (20) yield

$$\phi_n(t) \leqq \sum_{i=0}^{n} \frac{(2n)!(\phi_1(0))^i}{(2i)!2^{n-i}}\beta^{n-i}e^{2n\beta t}Q(t,n-i),$$

where

$$Q(t,0) = 1$$
$$Q(t,j) = \int_0^t Q(\zeta,j-1)e^{-2\beta\lambda}d\zeta$$
$$= \left(\frac{1}{2\beta}\right)^j\frac{(1-e^{-2\beta t})^j}{j!}.$$

Therefore

$$\phi_n(t) \leqq \sum_{i=0}^{n} \frac{(\phi_1(0))^i(2n)!(1-e^{-\beta t})^{n-i}}{(2i)!4(n - {}^{n-i}i)!},$$

and, since $(2i)! \geqq 2^i i!$,

$$\phi_n(t) \leqq \frac{(2n)!}{n!2^n} \sum_{i=0}^{n} \frac{(\phi_1(0))^i(1-e^{-\beta t})^{n-i}n!}{i!(n-i)!2^{n-i}}$$

$$= \frac{(2n)!}{n!2^n}\left(\phi_1(0) + \frac{1-e^{-\beta t}}{2}\right)^n.$$

This proves (16a), and (16b) follows from the fact that for ζ Gaussian with zero expectation

$$E\zeta^{2n} = \frac{(2n)!}{n!2^n}(E\zeta^2)^n.$$

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